Abstract

It is shown that the Weyl basis formed by the canonical symmetrization of an n-dimensional, p-rank tensor space with canonical projection operators of S_p is a Gel'fand basis of U(n). This basis may easily be generated using standard projection operator techniques.

Canonical symmetrization for unitary bases. I. Canonical Weyl bases*

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INTRODUCTION

It is well known that an n-dimensional, p-rank tensor space forms a reducible basis for both $U(n)$, the unitary group of n dimensions, and S_p, the symmetric group of p objects. The irreducible bases of the symmetric group formed from this tensor space (the symmetrized bases) can be used to reduce the bases of $U(n)$ and $SU(n)$, the unimodular subgroup of $U(n)$. The resultant irreducible bases of $U(n)$ and $SU(n)$ are called Weyl bases. Weyl generally avoided the explicit construction of these bases while using them to enumerate the bases of the irreducible representations (IR's) of $U(n)$ and to determine the characters of these IR's.[1]

However, works by Biedenharn, Baird, Ciftan, and Louck[2–6] have shown the need for an explicit construction of the irreducible bases of $U(n)$ in order to find the matrix elements of the IR's of $U(n)$ and the Clebsch–Gordon coefficients associated with the direct products of these IR's. Up to now this has been accomplished using a boson basis developed by Schwinger[7] for U(2) and extended to $U(n)$ by Biedenharn and Louck.[8] For certain applications this basis is more complicated than necessary. For example, work is presently being done in atomic and molecular orbital theory using a unitary basis to describe complex atomic and molecular configurations in a way that simplifies the evaluation of matrix elements of spin and orbital operators.[9–11] The explicit construction of a Weyl basis for $U(n)$ using the symmetrization techniques described in this paper has been instrumental in this simplification. A number of tableau "tricks" have appeared which make the use of a Weyl basis much more elegant and convenient. Already, the theory of permutation operators acting on a canonical Weyl basis has led to tableau algorithms for directly relating Slater determinant states to those states having separate spin and orbital parts and definite total angular momentum.[12] We expect more applications will be found in the future.

Attempts to construct a Weyl basis have been made in the past, notably by Littlewood.[13] Recent work has been done by Tompkins,[14] Sullivan,[15] and Lezuo.[16] Lezuo has succeeded in constructing a canonical Weyl basis for SU(2) and SU(3). His techniques have been simplified and extended in this work to yield a canonical Weyl basis for all $U(n)$ and $SU(n)$ as he himself had predicted. The success of the present method is a result of the choice of projection operators of S_p used to symmetrize the tensor space. Previous dependence upon Young symmetrizers to project out bases, as outlined by Hammermesh,[17] led to difficulties in the orthogonalization of the bases. These difficulties even created some doubt as to the possibility of ever constructing Weyl bases with projection operators.[18] The present choice of *canonical* projection operators of S_p not only makes orthogonality of the bases automatic, but also creates a *canonical* basis of $U(n)$ or a Gel'fand basis.

In Sec. I we review the relationship between the canonical irreducible bases of $U(n)$ and S_p. In particular, we show that the canonical bases of $U(n)$ are specified by the eigenvalues of the complete set of Gel'fand operators I_r^l for $r=1,2,\ldots,l$ and $l=1,2,\ldots,n$. Similarly, the canonical bases of S_p are specified by the eigenvalues of the complete set of r-cycle class operators K_r^l for $r=1,2,\ldots,l$ and $l=1,2,\ldots,p$.

In Sec. II we show that the Gel'fand invariant operators of $U(n)$ may be expanded in terms of the r-cycle class operators. The canonically symmetrized tensor basis transforms like a canonical basis of S_p under two different mutually commuting permutation operators which we call the particle and state permutations. In Sec. III we canonically symmetrize our tensor basis with *particle* projection operators of S_p. When acting on this basis, we show that the Gel'fand operators may be expanded in terms of the *state* r-cycle class operators of S_p. Furthermore, our symmetrized basis is an eigenfunction of all these *state* r-cycle class operators. Thus, we form canonical Weyl bases of $U(n)$ or Gel'fand bases from a canonically symmetrized tensor space. Since the canonical invariant operators of $SU(n)$ depend on the invariant operators of $U(n)$, we also form canonical bases of $SU(n)$.

In the following work (Paper II), we shall show that the boson basis is a generalization of the Weyl basis and, as a result, may be generated simply and straightforwardly using symmetrization operators. The boson basis is frequently called a "Weyl basis" even though it is constructed from a different tensor space than Weyl originally considered. Since we are constructing a basis from the same tensor space used by Weyl which may be used to generate a boson basis, we feel justified in adopting the name "Weyl basis."

I. REVIEW

A. Canonical irreducible bases of S_p

The symmetric group S_p consists of all $p!$ permutations of p objects. The IR's of S_p are labeled by the partitions $[u]=[u_1u_2\cdots u_p]$,[19] where $u_i \geq u_{i+1}$, $u_i > 0$ and $\sum_{i=1}^{p} u_i = p$. There are u_i boxes in row i of the partition $[u]$.

The matrix elements of the IR's of S_p can always be put in real form so that the IR's are orthogonal matrices. The IR's of S_p obey the standard orthonormality and completeness relations of group theory:

$$\frac{l^{[u]}}{p!}\sum_{i,j,[u]} D^{[u]}_{ij}(q)\,D^{[u]}_{ij}(q') = \delta_{(q)(q')}, \tag{1.1}$$

$$\frac{l^{[u]}}{p!}\sum_{q\in S_p} D^{[u]}_{ij}(q)\,D^{[u']}_{i'j'}(q) = \delta_{ii'}\,\delta_{jj'}\,\delta_{[u][u']}, \tag{1.2}$$

where $l^{[u]}$ is the dimension of the IR $[u]$ of S_p.

Permutations with the same cyclic structure belong to the same class of S_p. For the Lth class consisting of l_1 one-cycles, l_2 two-cycles, ..., and l_p p-cycles where $\sum_{i=1}^{p} i(l_i)=p$ we write $K_L = K_{1^{l_1}2^{l_2}\cdots p^{l_p}}$. Defining N_L to be the order of the Lth class, we have[20]

$$N_L = \frac{p!}{1^{l_1}l_1!\,2^{l_2}l_2!\cdots p^{l_p}l_p!}. \tag{1.3}$$

The character of K_L is defined by

$$\chi^{[u]}_L = \mathrm{Tr}D^{[u]}(q) \tag{1.4}$$

for all $q\in K_L$. A special case of (1.4) is

$$\chi^{[u]}_{1^p} = l^{[u]} = \mathrm{Tr}D^{[u]}(1). \tag{1.5}$$

A method of finding the dimension of IR $[u]$ using hook-lengths[21] is shown in Fig. 1.

The characters of S_p obey the standard orthonormality and completeness relations of group theory:

$$\frac{1}{p!}\sum_{[u]} N_L\,\chi^{[u]}_L\chi^{[u]}_{L'} = \delta_{LL'}, \tag{1.6}$$

$$\frac{1}{p!}\sum_{L} N_L\chi^{[u]}_L\chi^{[u']}_L = \delta_{[u][u']}. \tag{1.7}$$

We now define the projection operators of S_p by the relation:

$$P^{[u]}_{ij} = \frac{l^{[u]}}{p!}\sum_{t\in S_p} D^{[u]}_{ij}(t)\ (t). \tag{1.8}$$

Then it follows that

$$\begin{aligned}(q)\,P^{[u]}_{ij} &= \frac{l^{[u]}}{p!}\sum_{t\in S_p} D^{[u]}_{ij}(t)\ (q)(t),\\ &= \frac{l^{[u]}}{p!}\sum_{q'\in S_p} D^{[u]}_{ij}(q^{-1}q')\ (q'),\\ &= \frac{l^{[u]}}{p!}\sum_{i'}\sum_{q'\in S_p} D^{[u]}_{ii'}(q^{-1})\,D^{[u]}_{i'j}(q')\ (q').\end{aligned}$$

Using the orthogonality of the matrices,

$$D^{[u]}_{ii'}(q^{-1}) = D^{[u]}_{i'i}(q), \tag{1.9}$$

we have

$$(q)\,P^{[u]}_{ij} = \sum_{i'} P^{[u]}_{i'j}\,D^{[u]}_{i'i}(q). \tag{1.10}$$

Similarly,

$$P^{[u]}_{ij}(q) = \sum_{j'} P^{[u]}_{ij'}\,D^{[u]}_{jj'}(q). \tag{1.11}$$

Furthermore, the projection operators are orthogonal as seen by using (1.2) and (1.10),

$$\begin{aligned}P^{[u]}_{ij}P^{[v]}_{kl} &= \frac{l^{[u]}}{p!}\sum_{q\in S_p} D^{[u]}_{ij}(q)\ (q)P^{[v]}_{kl}\\ &= \delta_{jk}\delta_{[u][v]}\,P^{[v]}_{il}.\end{aligned} \tag{1.12}$$

Now consider the following idempotent:

$$P^{[u]} = \sum_i P^{[u]}_{ii}. \tag{1.13}$$

We may expand $P^{[u]}$ in terms of the classes of S_p. We have

$$\begin{aligned}P^{[u]} &= \frac{l^{[u]}}{p!}\sum_i\sum_{q\in S_p} D^{[u]}_{ii}(q)\ (q)\\ &= \frac{l^{[u]}}{p!}\sum_{q\in S_p} \mathrm{Tr}D^{[u]}(q)\ (q),\end{aligned}$$

or

$$P^{[u]} = \frac{l^{[u]}}{p!}\sum_L \chi^{[u]}_L K_L. \tag{1.14}$$

We may also expand the classes of S_p in terms of the idempotents. Using (1.14) and (1.6) we have

$$\frac{1}{p!}\sum_{[u]}\sum_L N_{L'}\chi^{[u]}_{L'}\chi^{[u]}_L K_L = \sum_{[u]}\frac{N_{L'}\chi^{[u]}_{L'}P^{[u]}}{l^{[u]}},$$

or

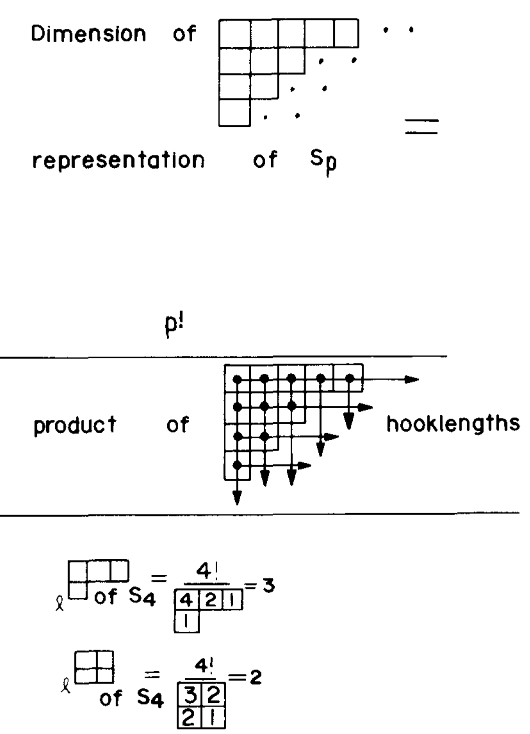

FIG. 1. Dimension of an IR of S_p. The Robinson formula for the dimension $l^{[u]}$ of an IR of S_p labeled by a partition $[u]$ is given above. A hook-length of a box in a partition is simply the number of boxes below it, to the right of it, and itself. The denominator in the dimension formula is the numerical product of all the hook-lengths of the boxes. Examples for S_4 are given below.

$$K_L = \sum_{[u]} \frac{N_L \chi_L^{[u]} P^{[u]}}{l^{[u]}}. \tag{1.15}$$

We can now show that the irreducible bases of S_p are eigenvectors of the classes of S_p. Let $|{}^{[u]}_{i}\rangle$ be an irreducible basis of the IR $[u]$ of S_p. By definition we have

$$P^{[v]} |{}^{[u]}_{i}\rangle = \sum_{i'} |{}^{[u]}_{i'}\rangle D^{[u]}_{i'i}(P^{[v]}).$$

From (1.13), (1.2), and (1.8) it follows that

$$P^{[v]} |{}^{[u]}_{i}\rangle = |{}^{[u]}_{i}\rangle \delta_{[u][v]}. \tag{1.16}$$

Operating on the irreducible basis with class K_L and using (1.15) and (1.16), we have

$$K_L |{}^{[u]}_{i}\rangle = \frac{N_L \chi_L^{[u]}}{l^{[u]}} |{}^{[u]}_{i}\rangle, \tag{1.17}$$

so the bases of the IR $[u]$ of S_p are simultaneous eigenvectors of the class operators of S_p with eigenvalues $(N_L \chi_L^{[u]})/l^{[u]}$.

The simultaneous eigenvalues of the class operators completely determine the IR's of S_p. If

$$\frac{N_L \chi_L^{[u]}}{l^{[u]}} = \frac{N_L \chi_L^{[v]}}{l^{[v]}}$$

for all L, then

$$\frac{1}{p!} \sum_L \frac{N_L \chi_L^{[u]} \chi_L^{[u]}}{(l^{[u]})^2} = \frac{1}{p!} \sum_L \frac{N_L \chi_L^{[v]} \chi_L^{[v]}}{(l^{[v]})^2},$$

and from (1.7) it follows that $l^{[u]} = l^{[v]}$. From our initial assumption we have that $\chi_L^{[u]} = \chi_L^{[v]}$ for all L. This is precisely the criteria for the two IR's $[u]$ and $[v]$ of S_p to be equivalent. However, since we only need to specify the p rows $u_1, u_2, \ldots, u_p$ to determine the IR $[u]$ of S_p, not all of the class operators are independent. Kramer[22] has shown that the eigenvalues of the r-cycle class operator, $K_r \equiv K_{1^{p-r}r^1}$ for $r = 2, 3, \ldots, p$, uniquely determine the IR's of S_p. Note that we need the $p-1$ independent operators K_r and the condition, $p = \sum_{i=1}^{p} u_i$, to determine the p unknowns $u_1, u_2, \ldots, u_p$. For this reason we adjoin to the r-cycle class operators the operator $K_1 = p$, and adopt the notation

$$K_r^p = K_r \quad \text{for } r = 2, 3, \ldots, p,$$

$$K_1^p = K_1 = p.$$

This notation makes explicit the fact that the eigenvalue p of K_1^p determines the group S_p to which the r-cycle class operators belong.

A canonical irreducible basis of S_p is defined by means of the subgroup reduction $S_p \supset S_{p-1} \supset \cdots \supset S_1$ such that an irreducible canonical basis of S_p is also an irreducible basis for all subgroups in this reduction. Thus a canonical irreducible basis of S_p is an eigenvector of all sets of class operators of $S_p, S_{p-1}, \ldots, S_1$:

$$\begin{matrix} K_p^p \; K_{p-1}^p \cdots K_1^p \\ K_{p-1}^{p-1} \cdots K_1^{p-1} \\ \ddots \quad \vdots \\ K_1^1 \end{matrix}.$$

The lth row of r-cycle class operators above completely specifies an IR $[u]^l$ of S_l so that a canonical irreducible basis $|{}^{[u]}_{(r)}\rangle$ of S_p may be represented by means of a succession of partitions:

$$ {}^{[u]}_{(r)} = \begin{pmatrix} [u] \\ [u]_r^{p-1} \\ \vdots \\ [u]_r^1 \end{pmatrix} = \begin{pmatrix} [u_{1p} u_{2p} \cdots u_{pp}] \\ [u_{1p-1} \cdots u_{p-1\,p-1}] \\ \vdots \\ [u_{11}] \end{pmatrix}, \tag{1.18}$$

where $\sum_{i=1}^{m} u_{im} = m$ for $m = 1, 2, \ldots, p$, and $[u]^p = [u]$. We call such a succession of partitions a standard pattern of S_p. The fact that this pattern uniquely determines a canonical irreducible basis of S_p is inherent in the branching law of S_p which specifies further that[23]

$$u_{i\,m+1} \geq u_{im} \geq u_{i+1\,m+1}. \tag{1.19}$$

The standard pattern of S_p may be pictured by means of a standard tableau[24] $T^{[u]}_{(r)}$ of S_p. The standard tableau $T^{[u]}_{(r)}$ is the partition $[u]$ of S_p numbered with $1, 2, \ldots, p$ in the boxes such that the numbers increase to the right in the rows and down in the columns with no numbers repeated. There are $l^{[u]}$ such standard tableaus of $[u]$. For example, when $[u] = [210]$, then $l^{[u]} = 2$, and we have the two standard tableaus $^{12}_{3}$ and $^{13}_{2}$. For typographic convenience we have omitted the boxes surrounding the numbers in the standard tableaus.

Let $T^{[u]^m}_{(r)}$ be the standard tableau remaining after removing boxes with numbers $m+1, m+2, \ldots, p$ from $T^{[u]}_{(r)}$ and let $[u]_r^m$ be the corresponding partition remaining. The rows u_{im} of partition $[u]_r^m$ obey conditions (1.19) of the branching law of S_p so that we may associate the standard tableau $T^{[u]}_{(r)}$ with the standard pattern ${}^{[u]}_{(r)}$ of S_p. The standard tableau associated with a canonical irreducible basis uniquely determines the reduction of the basis under all subgroups $S_p, S_{p-1}, \ldots, S_1$ as illustrated in Fig. 2.

We now give a prescription for finding the seminormal canonical projection operators $O^{[u]}_{rs}$ of S_p defined such that

$$O^{[u]}_{rs} = C^{[u]}_{rs} P^{[u]}_{rs} \tag{1.20}$$

where $C^{[u]}_{rs}$ is some positive constant and $P^{[u]}_{rs}$ is a canonical projection operator constructed from a canonical IR of S_p.[25] We will show later that the $O^{[u]}_{rs}$ can project out a canonical irreducible basis of S_p. Let $T^{[u]}_{(r)} = \sigma_{rs} T^{[u]}_{(s)}$ where σ_{rs} permutes the numbers in the standard tableau $T^{[u]}_{(r)}$ of S_p. For example,

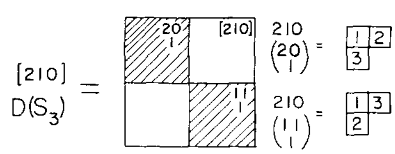

FIG. 2. Canonical reduction for S_p. The IR [210] of S_3 reduces to the IR's [20] and [11] of S_2 as shown. As a representation of S_2 or S_1 it is diagonal. Each basis of [210] corresponds to a diagonal component with a unique genealogy traced by the standard patterns or standard tableaus shown on the right.

$\begin{smallmatrix}12\\3\end{smallmatrix} = (23)\begin{smallmatrix}13\\2\end{smallmatrix}$.

Now let P_r^m be the symmetrization operator of the numbers in the rows of $\mathbf{T}_{(r)}^{[u]^m}$ and N_r^m be the antisymmetrization operator of the numbers in the columns of $\mathbf{T}_{(r)}^{[u]^m}$. In the example above we have

$$P^3_{\substack{12\\3}} = S_{12} = (1) + (12),$$

$$N^3_{\substack{12\\3}} = A_{12} = (1) - (13).$$

If we define

$$E_{rs}^p = P_r^p \sigma_{rs} N_s^p. \quad (1.21)$$

then

$$O_{rs}^{[u]} = O_{rs}^{[u]^p} = O_{rr}^{[u]^{p-1}} E_{rs}^p O_{ss}^{[u]^{p-1}}, \quad (1.22)$$

where $O^{[u]^1} = (1)$. It is interesting to note that we obtain the same $O_{rs}^{[u]}$ if we define $E_{rs}^p = N_r^p \sigma_{rs} P_s^p$. Continuing with our example, we have $E^2_{12\,12} = S_{12}$, $E^2_{\substack{13\\2}\substack{13\\2}} = A_{12}$, and $E^3_{\substack{12\\3}\substack{13\\2}} = S_{12}(23)A_{12}$ so that

$$\begin{aligned} O^{[210]}_{\substack{12\\3}\substack{13\\2}} &= S_{12}S_{12}(23)A_{12}A_{12} \\ &= 4S_{12}(23)A_{12} \\ &= 4((1)+(12))(23)((1)-(12)) \\ &= 4((23)-(13)+(132)-(123)). \end{aligned}$$

Let $[\tilde{u}]$ be the partition conjugate to $[u]$ formed by exchanging rows and columns of $[u]$. Similarly, $\mathbf{T}_{(\tilde{r})}^{[\tilde{u}]}$ is the conjugate standard tableau formed by exchanging rows and columns of $\mathbf{T}_{(r)}^{[u]}$. For example,

$$\widetilde{\begin{smallmatrix}123\\4\end{smallmatrix}} = \begin{smallmatrix}14\\2\\3\end{smallmatrix}.$$

Also let ϵ_q be 1 or -1 if permutation (q) is an even or odd product of bicycles respectively. We see that $\epsilon_{\sigma_{rs}} = \epsilon_{\sigma_{\tilde{r}\tilde{s}}}$ and that $O_{\tilde{r}\tilde{s}}^{[\tilde{u}]}$ may be found from $O_{rs}^{[u]}$ simply by exchanging symmetrization and antisymmetrization operators ($A \leftrightarrow S$). It follows that the coefficient of (q) in $O_{\tilde{r}\tilde{s}}^{[\tilde{u}]}$ is simply the coefficient of (q) in $O_{rs}^{[u]}$ multiplied by the factor $\epsilon_q \epsilon_{\sigma_{rs}}$. From (1.20) and the definition of the projection operators (1.8), we have the relation

$$D_{\tilde{r}\tilde{s}}^{[\tilde{u}]}(q) = \epsilon_{\sigma_{rs}} \epsilon_q D_{rs}^{[u]}(q) \quad (1.23)$$

for the canonical IR's of S_p.

B. Canonical irreducible bases of U(n) and SU(n)

The unitary group U(n) consists of the set of all n-dimensional unitary matrices. The unimodular unitary group SU(n) is the set of all n-dimensional unitary matrices with unimodular determinant (determinant equals one) and is a subgroup of U(n). The set U(n) and SU(n) form an IR of themselves which is called the self or fundamental representation.

The generators E_{ij} for $i, j = 1, 2, \ldots, n$ of U(n) obey the commutation relation[26]

$$[E_{ij}, E_{kl}] = E_{il}\delta_{jk} - E_{kj}\delta_{il}. \quad (1.24)$$

where

$$E_{ij}^\dagger = E_{ji}. \quad (1.25)$$

The generators E'_{ij} of SU(n) will obey the unimodular condition

$$\mathrm{Tr}E'_{ij} = 0, \quad (1.26)$$

if we let

$$E'_{ij} = E_{ij} \quad \text{for } i \neq j, \quad (1.27)$$

$$H_i = E'_{ii} = \frac{1}{\sqrt{i(i+1)}}(E_{11} + E_{22} + \cdots + E_{ii} - iE_{i+1\,i+1}) \quad \text{for } i = 1, 2, \ldots, n-1.$$

The IR's of U(n) and SU(n) are labeled by partitions $[M]$ with the number of rows no greater than n:[27]

$$[M] = [m_{1n}\, m_{2n} \cdots m_{nn}].$$

One of the first problems is to find a complete and independent set of mutually commuting Hermitian invariant operators constructed from the generators E_{ij} or E'_{ij} which uniquely specify the IR of U(n) or SU(n) respectively. By invariant operators we mean that the operators commute with all the generators of the group.

Such a set of invariant operators has been found by Gel'fand[28] for U(n):

$$I_k^n = \sum_{i_1, i_2, \ldots, i_k}^{n} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_k i_1} \quad (1.28)$$

$$\text{for } k = 2, 3, \ldots, n,$$

$$I_1^n = \sum_{i_1}^{n} E_{i_1 i_1}.$$

From (1.25) it follows that the I_k^n are Hermitian and from the commutation relations (1.24) it is straightforward to show that

$$[E_{ij}, I_k^n] = 0 \quad (1.29)$$

for $k = 1, 2, \ldots, n$. Thus the operators I_k^n are invariant and mutually commuting. It can also be shown that the operators I_k^n are independent and complete, i.e., the eigenvalues of these operators uniquely specify the IR $[M]$ of U(n) and the eigenvectors of these operators form an orthogonal irreducible basis of $[M]$. Note that n operators are necessary to specify the n rows of partition $[M]$.

For SU(n) the dependence of the invariant operators $I_k'^n$ for $k = 2, 3, \ldots, n$ upon the generators E'_{ij} is more complicated. Such a set of mutually commuting Hermitian invariant operators has been found by Popov and Perelomov[29]:

$$I_k'^n = \sum_{r=0}^{k} \binom{k}{r} \left(-\frac{I_1^n}{n}\right)^{k-r} I_r^n, \quad I_0^n = n. \quad (1.30)$$

This is one way to prove that an irreducible basis of U(n) is also an irreducible basis of SU(n). Hence, the bases of U(n) and SU(n) may be simultaneously specified by the same partition as we have indicated. There are only $n-1$ operators needed to specify the partition $[M]$, because for SU(n) we have the equivalence of IR's[30]:

$$[m_{1n}\, m_{2n} \cdots m_{nn}] = [m_{1n} - m_{nn}\;\; m_{2n} - m_{nn} \cdots m_{n-1n} - m_{nn}\; 0], \quad (1.31)$$

or

$$[M] = [M']$$

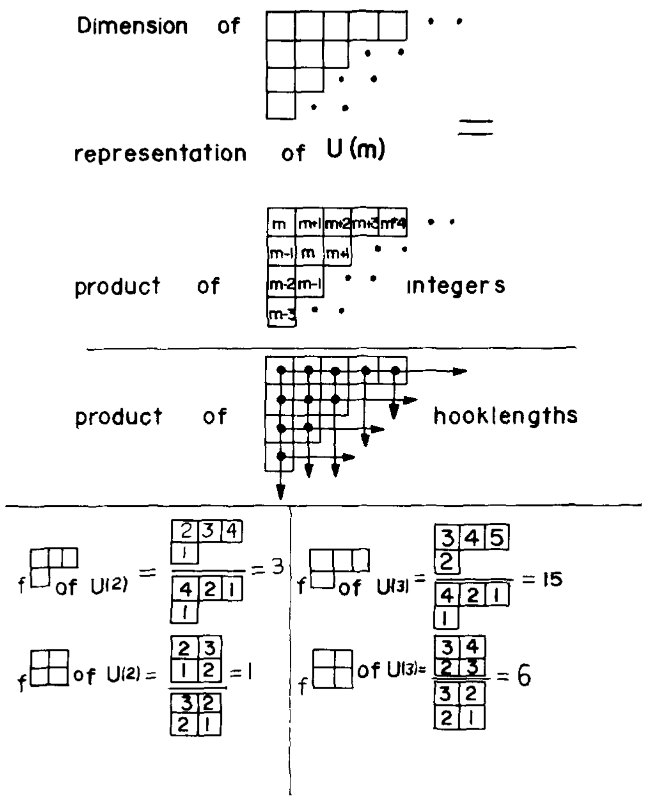

FIG. 3. Dimension of an IR of U(m). The Robinson formula for the dimension $f^{[M]}$ of an IR of U(m) labeled by a partition $[M]$ is given above. Examples for U(2) and U(3) are given below.

One may remove all the columns with n boxes from the partition and obtain the same IR of SU(n).

A canonical irreducible basis of U(n) is defined by means of the subgroup reduction U$(n) \supset$ U$(n-1) \supset \cdots \supset$ U(1) such that an irreducible canonical basis of U(n) is also an irreducible basis for all subgroups in this reduction. Thus, a canonical irreducible basis of U(n) is an eigenvector of all the sets of invariant operators of U(n), U$(n-1), \ldots,$ U(1):

$$
\begin{array}{l}
I_n^n I_{n-1}^n \cdots I_1^n \\
I_{n-1}^{n-1} \cdots I_1^{n-1} \\
\quad \ddots \quad \cdot \\
\qquad I_1^1 \quad .
\end{array}
$$

The lth row uniquely specifies an IR of U(l) so that a canonical irreducible basis $|{}^{[M]}_{\langle m\rangle}\rangle$ of U(n) may be represented by means of a succession of partitions

$$
{}^{[M]}_{\langle m\rangle} = \left\langle \begin{array}{c} [m_{1n}\, m_{2n} \cdots m_{nn}] \\ [m_{1n-1} \cdots m_{n-1\,n-1}] \\ \ddots \quad \cdot \\ [m_{11}] \end{array} \right\rangle . \qquad (1.32)
$$

The pattern above is called a Gel'fand pattern[31] or a standard pattern of U(n). The fact that this pattern uniquely determines a canonical irreducible basis of U(n) is inherent in the Weyl branching law[32] which specifies further that

$$
m_{i\,j+1} \geq m_{ij} \geq m_{i+1\,j+1}. \qquad (1.33)
$$

There are $f^{[M]}$ such patterns, where $f^{[M]}$ is the dimension of the IR $[M]$ of U(n). A method of finding the dimension $f^{[M]}$ using hook-lengths is shown in Fig. 3. If we let $\lambda_l = p_l - p_{l-1}$ where $p_l = \sum_{i=1}^{l} m_{il}$, then $(\lambda) = (\lambda_1 \lambda_2 \cdots \lambda_n)$ is called the weight of the standard pattern ${}^{[M]}_{\langle m\rangle}$.

The standard pattern of U(n) may be pictured by means of a standard tableau $T^{[M]}_{\langle m\rangle}$ of U(n). The standard tableau $T^{[M]}_{\langle m\rangle}$ is the partition $[M]$ of U(n) numbered with $1, 2, \ldots, n$ in the boxes such that the numbers are nondecreasing to the right in the rows and are increasing down in the columns. There may be repeated numbers in the rows but not in the columns. When there are no repeated numbers in the rows of $T^{[M]}_{\langle m\rangle}$ of U(n) it will be equivalent to some $T^{[M]}_{\langle m\rangle}$ of S_n. There are $f^{[M]}$ standard tableaus of U(n) corresponding to IR $[M]$. For example, when $[M]=[210]$, $f^{[M]}=8$, and we have the eight

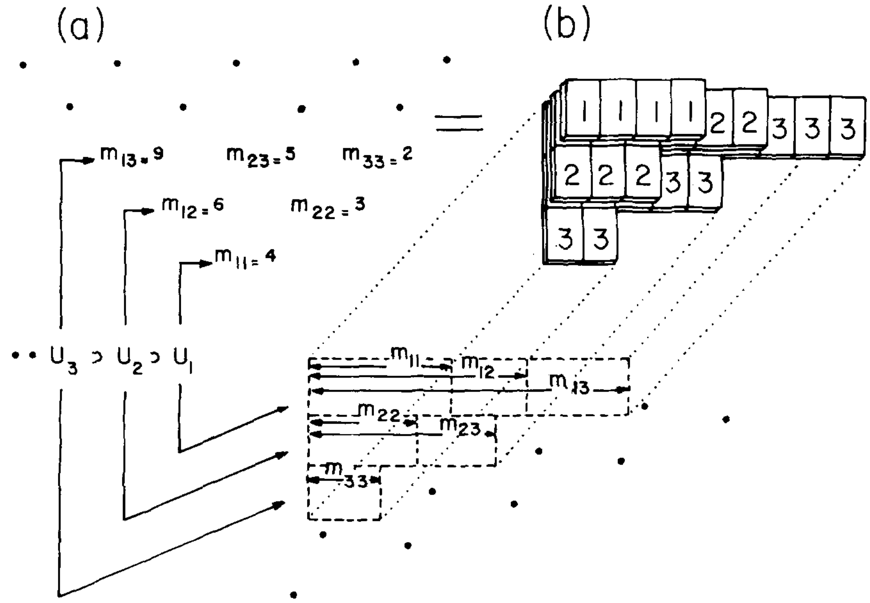

FIG. 4. Labeling for a canonical unitary basis. (a) Standard or Gel'fand pattern of U(n). The lth row of integers $[m_{1l} m_{2l} \cdots m_{ll}]$ specifies to which IR of U(l) the basis belongs. The $(l-1)$th row specifies to which IR of U$(l-1)$ the IR of U(l) reduces. In this way each basis of U(n) has a unique genealogy chain and labeling. (b) Standard tableau of U(n). The standard pattern of U(n) may be pictured by means of a standard tableau. (When labeled algebraically, it is just an upside down standard pattern.)

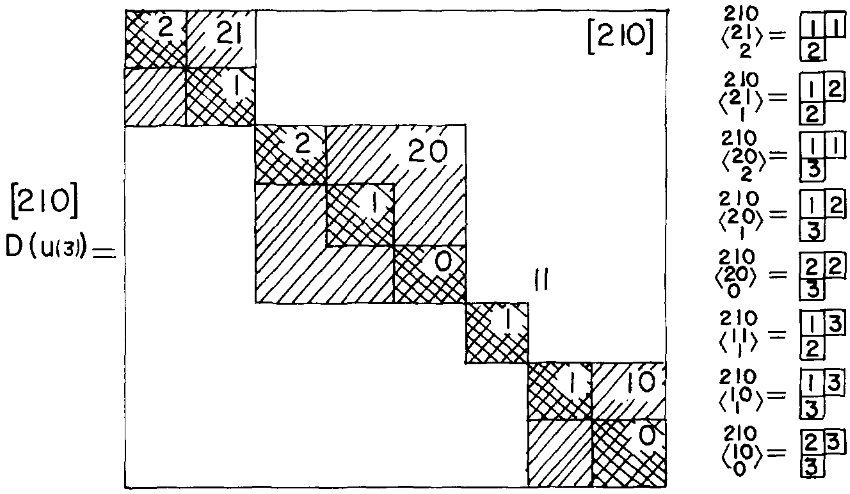

FIG. 5. Canonical reduction for U(n). The IR [210] of U(3) reduces to the IR's [21], [20], [11], and [10] of U(2) as shown. As a representation of U(1) it is diagonal. Each basis of [210] corresponds to a diagonal component with a unique genealogy traced by the standard patterns or standard tableaus shown on the right.

standard tableaus $\substack{11\\2}$, $\substack{12\\2}$, $\substack{11\\3}$, $\substack{12\\3}$, $\substack{22\\3}$, $\substack{13\\2}$, $\substack{13\\3}$, and $\substack{23\\3}$.

Let the standard tableau $T^{[M]}_{\langle m\rangle}$ contain λ_l numbers l and let $[M]^{p_l}_m$ be the partition remaining after removing boxes with the number $l+1, l+2, \dots, n$ from $T^{[M]}_{\langle m\rangle}$. We see that the rows m_{il} of the partition $[M]^{p_l}_m$ must obey conditions (1.33) of the branching law of U(n) so that we may associate the standard tableau $T^{[M]}_{\langle m\rangle}$ with the standard pattern $\substack{[M]\\ \langle m\rangle}$ as shown in Fig. 4. The standard tableau associated with a canonical irreducible basis uniquely determines the reduction of the basis under all subgroups U(n), U($n-1$), ..., U(1) as illustrated in Fig. 5.

It is impossible to completely specify an irreducible representation of SU(n) by means of the subgroup reduction SU(n) ⊃ SU($n-1$) ⊃ ⋯ ⊃ SU(1) as one might expect.[33] Instead one uses the canonical reduction SU(n) ⊃ $U_{n-1}(1)$ × SU($n-1$) ⊃ $U_{n-2}(1)$ × SU($n-2$) ⊃ ⋯ ⊃ $U_2(1)$ × SU(2) ⊃ $U_1(1)$, where the generator for $U_l(1)$ is H_l. Note that H_l commutes with all generators E_{ij} of SU(l) as required for the direct product $U_l(1)\times SU(l)$. H_l is also the only invariant operator of $U_l(1)$, so the complete set of invariant operators for $U_l(1)\times SU(l)$ is H_l and I'^{l}_k for $k=2,3,\dots,l$. A canonical irreducible basis of SU(n) is then an eigenvector of all the sets of invariant operators:

$$
\begin{array}{l}
I'^{n}_{n}\, I'^{n}_{n-1} \cdots I'^{n}_{2}\; 0 \\
I'^{n-1}_{n-1} \cdots I'^{n-1}_{2}\, H_{n-1} \\
\quad \vdots \qquad \vdots \\
\quad I'^{2}_{2}\; H_2 \\
\qquad H_1
\end{array}
$$

Because of the relations (1.27) and (1.30), a canonical irreducible basis of SU(n) is specified by the simultaneous eigenvalues of the invariant operators of U(n) within the equivalence

$$\left|\substack{[M]\\ \langle m\rangle}\right\rangle = \left|\substack{[M']\\ \langle m\rangle}\right\rangle. \tag{1.34}$$

II. U(n) INVARIANT OPERATORS AND S_p CLASS OPERATORS

A. Unitary invariant operators

If $f(h_1, h_2, \dots, h_n)$ is any symmetric polynomial function of its arguments $(h)=(h_1,h_2,\dots,h_n)$, and $(p)(h) = (h_p)$, where $(p)\in S_n$, then from the definition of a symmetric function we have

$$(p)f(h) = f(h_p) = f(h). \tag{2.1}$$

It has been shown by Perelomov and Popov[34] that the eigenvalues of the invariant operators of U(n), $\langle I^n_k\rangle$, for $k=1,2,\dots,n$, are symmetric kth degree polynomials of the partial hooks,

$$h_i = u_i + n - i \tag{2.2}$$

for $i=1,2,\dots,n$, where $[u]=[u_1u_2\cdots u_n]$ is any IR of U(n). Furthermore, the only kth degree terms of the (h) in $\langle I^n_k\rangle$ are

$$S_k = \sum_{i=1}^{n} h_i^k. \tag{2.3}$$

The S_k for $k=1,2,\dots,n$ are the Pythagorian symmetric functions. From the fundamental theorem of symmetric functions[35] any symmetric kth degree polynomial function of (h) for $k\le n$ is expressible as a polynomial function of the $S_1, S_2, \dots, S_k$ of kth degree in (h). A symmetric kth degree polynomial function of (h) for $k>n$ is expressible as a polynomial function of only the $S_1, S_2, \dots, S_n$ of nth degree in (h). Clearly, from the above we have

$$\langle I^n_k\rangle = S_k + F^n_{k-1}(S_1, S_2, \dots, S_{k-1}), \tag{2.4}$$

for $k=1,2,\dots,n$, where F^n_{k-1} is a polynomial of the $S_1, S_2, \dots, S_{k-1}$ of degree $k-1$ in (h) and F_0 is a constant.

We can now prove that the invariant operators of U(n) are independent and complete, and therefore uniquely specify the IR $[u]$ of U(n). To prove the independence of the I^n_k for $k=1,2,\dots,n$ we show that the Jacobian,

$$J(\langle I^n_k\rangle) = \frac{\partial(\langle I^n_1\rangle\langle I^n_2\rangle\cdots\langle I^n_n\rangle)}{\partial(u_1u_2\cdots u_n)}, \tag{2.5}$$

is nonvanishing.[36] From (2.4) we have

$$J(\langle I^n_k\rangle) = J(S^n_k),$$

$$= n! \begin{vmatrix} h_1^{n-1} & h_1^{n-2} & \cdots & 1 \\ h_2^{n-1} & h_2^{n-2} & \cdots & 1 \\ \vdots & & & \vdots \\ h_n^{n-1} & h_n^{n-2} & \cdots & 1 \end{vmatrix}$$

$$= n!\, D(h_1, h_2, \ldots, h_n), \tag{2.6}$$

where

$$D(h_1, h_2, \ldots, h_n) = \prod_{i<j}^{n} (h_i - h_j)$$

is the Vandermonde determinant. Now $h_i \neq h_j$ for any $i \neq j$ since $u_i - i \neq u_j - j$ for any partition $[u]$. So $J(\langle I_k^n \rangle)$ is nonvanishing and the operators I_k^n for $k = 1, 2, \ldots, n$ are independent.

To prove completeness,[37] we must show that

$$h_i = h_i(\langle I_1^n \rangle, \langle I_2^n \rangle, \ldots, \langle I_n^n \rangle) \tag{2.7}$$

for $i = 1, 2, \ldots, n$. Then any invariant operator I^n where $\langle I^n \rangle = I^n(h_1, h_2, \ldots, h_n)$ may be expressed in terms of $I_1^n, I_2^n, \ldots, I_n^n$ using (2.7). This is equivalent to showing that the n independent equations,

$$\langle I_k^n \rangle = I_k^n(h_1, h_2, \ldots, h_n) \tag{2.8}$$

for $k = 1, 2, \ldots, n$ have only one solution (h) which corresponds to a partition.

Let (h) be such a solution of (2.8) corresponding to partition $[u]$. Then (h_p) is also a solution of (2.8) since the $I_k^n(h_1, h_2, \ldots, h_n)$ are symmetric functions of (h). Also, since $h_i \neq h_j$ for $i \neq j$ we have $n!$ distinct solutions (h_p) for all $p \in S_n$. But this is the maximum number of distinct solutions allowed from the n equations of (2.8) since $I_k^n(h_1, h_2, \ldots, h_n)$ is of kth degree in (h) and

$$\prod_{k=1}^{n} (k) = n!.$$

So the (h_p) give us all possible solutions of (2.8).

Now define $[u'] < [u]$ if the first nonzero difference $u_i - u_i'$ for $i = 1, 2, \ldots, n$ is positive. If $[u]$ is a partition then $(p)[u] \leq [u]$. Let $R_i = n - i$ and $(R) = (R_1, R_2, \ldots, R_n)$ so that $(h) = [u] + (R)$. We note that $(p)(R) < (R)$ and $-(R) < -(p^{-1})(R)$ for $(p) \neq (1)$. Now let $[u_p] = (h_p) - (R)$ be the partition corresponding to the solution (h_p) of (2.8). Then for $(p) \neq (1)$

$$\begin{aligned} [u_p] &= (p)[u] + (p)(R) - (R) \\ &< [u] + (R) - (p^{-1})(R) = (p^{-1})[u_p]. \end{aligned} \tag{2.9}$$

So $[u_p]$ cannot be a partition unless $(p) = (1)$. We have only one solution (h) which corresponds to a partition.

B. Permutation class operators

For a number of applications in theoretical spectroscopy[38] it is convenient to recast some of the unitary operator formalism in terms of r-cycle permutation operators. In particular, we will show that the invariant operator eigenvalues of U(n) and SU(n) can be expressed in terms of the r-cycle class operator eigenvalues given in (1.17). This is convenient because the eigenvalues of the K_r are easily evaluated in terms of a hook-length formula given in Appendix A.

Let us consider the eigenvalues of the K_r of S_p in more detail. If we restrict our attention to partitions $[u] = [u_1 u_2 \cdots u_n]$ with n rows, it has been shown by Yamanouchi that[39]

$$\chi_r^{[u]} = (p-r)! \sum_{i=1}^{n} \frac{D(h_1, h_2, \ldots, h_i - r, \ldots, h_n)}{h_1! h_2! \cdots (h_i - r)! \cdots h_n!}, \tag{2.10}$$

for $r = 0, 2, 3, \ldots, n$, where the sum is over all indices such that there are no negative factorials. From (2.10) it follows that for $r = 0$,

$$\chi_{1^p}^{[u]} = l^{[u]} = \frac{p!}{h_1! h_2! \cdots h_n!} D(h_1, h_2, \ldots, h_n), \tag{2.11}$$

and from (1.3) and (1.17) that

$$\langle K_r \rangle = \frac{1}{r} \sum_{i=1}^{n} \frac{h_i!\, D(h_1, h_2, \ldots, h_i - r, \ldots, h_n)}{(h_i - r)!\, D(h_1, h_2, \ldots, h_n)}, \tag{2.12}$$

for $r = 2, 3, \ldots, n$.

It can now be seen that the eigenvalues of the operators K_r are symmetric rth degree polynomials of the partial hooks (h). Thus $\langle K_r \rangle$ for $r = 2, 3, \ldots, n$ is a polynomial function of the Pythagorian symmetric functions $S_1, S_2, \ldots, S_r$ when K_r acts on an irreducible bases of $[u]$ with n rows.

We would like to find the class operators which are complete and independent when acting on such a basis. For this purpose we adjoin to the r-cycle class operators the invariant operator I_1^n of U(n), where

$$\langle I_1^n \rangle = \sum_{i=1}^{n} u_i = p_n = p$$

and adopt the notation:

$$\begin{aligned} K_r^{p_n} &= K_r \quad \text{for } r = 2, 3, \ldots, n, \\ K_1^{p_n} &= I_1^n. \end{aligned} \tag{2.13}$$

This notation makes explicit the fact that the eigenvalue p_n of I_1^n determines the group S_{p_n} to which the r-cycle class operators belong.

We may expand the $\langle K_r^{p_n} \rangle$ in terms of polynomials of S_r of rth degree in (h):

$$\langle K_r^{p_n} \rangle = \frac{1}{r} \sum_{k=1}^{r} \sum_{(v)} a_k^{(v)} (S_1^{v_1} S_2^{v_2} \cdots S_k^{v_k}), \tag{2.14}$$

where we sum over those $(v) = (v_1, v_2, \ldots, v_k)$ such that $\sum_{i=1}^{k} i(v_i) = k$. We know that

$$\langle K_1^{p_n} \rangle = \sum_{i=1}^{n} u_i = S_1 + n(n-1)/2. \tag{2.15}$$

We wish to prove that

$$\langle K_r^{p_n} \rangle = (1/r) S_r + f_{r-1}^n (S_1, S_2, \ldots, S_{r-1}), \tag{2.16}$$

for $r = 2, 3, \ldots, n$, where f_{r-1}^n is a polynomial of the $S_1, S_2, \ldots, S_{r-1}$ of degree $r-1$ in (h). This is equivalent to proving that the only rth degree terms of the (h) in $\langle K_r^{p_n} \rangle$ are $(1/r) S_r$. We follow closely the procedure of Hammermesh.[40]

From (2.12) and (2.14) we have

$$\begin{aligned} &\sum_{i=1}^{n} \frac{h_i!}{(h_i - r)!} D(h_1, h_2, \ldots, h_i - r, \ldots, h_n) \\ &= D(h_1, h_2, \ldots, h_n) \sum_{k, (v)} a_k^{(v)} (S_1^{v_1} S_2^{v_2} \cdots S_k^{v_k}). \end{aligned} \tag{2.17}$$

To find the $a_k^{(v)}$, we equate coefficients of like monomials of (h) on both sides of Eq. (2.17). Consider only terms in which the power of h_1 is $\geq n$ and in which the power of all other h_i is $\geq n-r$. Then

$$\frac{h_1!}{(h_1-r)!}\begin{vmatrix} (h_1-r)^{n-1} & (h_1-r)^{n-2} & \cdots & (h_1-r)^{n-r} \\ h_2^{n-1} & h_2^{n-2} & \cdots & h_2^{n-r} \\ \vdots & & & \vdots \\ h_r^{n-1} & h_r^{n-2} & \cdots & h_r^{n-r} \end{vmatrix} |h_{r+1}^{n-r-1}\cdots h_{n-1}|$$

$$= \begin{vmatrix} h_1^{n-1} & h_1^{n-2} & \cdots & h_1^{n-r} \\ h_2^{n-1} & h_2^{n-2} & \cdots & h_2^{n-r} \\ \vdots & & & \vdots \\ h_r^{n-1} & h_r^{n-2} & \cdots & h_r^{n-r} \end{vmatrix} |h_{r+1}^{n-r-1}\cdots h_{n-1}|$$

$$\times \sum_{k,(v)} a_k^{(v)} S_1^{v_1} S_2^{v_2}\cdots S_k^{v_k}),$$

where $|h_{r+1}^{n-r-1}\cdots h_{n-1}|$ is the determinant left after removing the first r rows and columns from $D(h_1, h_2, \ldots, h_n)$. Equating the coefficients of the monomials of highest degree on both sides, we find

$$a_r^{(v)} = \begin{cases} 1 & \text{when } (v) = (00\cdots 01), \\ 0 & \text{when } (v) \neq (00\cdots 01). \end{cases} \tag{2.18}$$

The remaining terms of $\langle K_r^{p_n}\rangle$ must be symmetric polynomials of (h) of degree less than r, thus proving (2.16).

We may prove, as we did for the invariant operators I_r^n of U(n), that the r-cycle class operators are complete and independent when operating on irreducible bases of $[u]$ with n rows. The operator $K_1^{p_n}$ uniquely specifies the permutation group S_{p_n} to which the r-cycle class operators belong. The remaining $K_r^{p_n}$ for $r = 2, 3, \ldots, n$ then uniquely specify the partition $[u]$ of S_{p_n} with n rows.

As a corollary to the above, let $[u]$ be any partition with p boxes so that it labels an IR of S_p. Since such a partition can not have over p rows it must be specified by the operators K_r^p for $r = 1, 2, \ldots, p$, where $K_1^p = p$. This proves the results of Kramer given in Sec. I.

We have assumed that the IR $[u]$ of U(n) has n rows. Now let the IR $[u]$ of U(n) have only k rows, that is, we assume $u_{k+1} = u_{k+2} = \cdots = u_n = 0$. Then $\langle I_r^n\rangle$ and $\langle K_r^{p_n}\rangle$ may be expanded as a polynomial function of the $S_1, S_2, \ldots, S_k$ of kth degree in $(h) = (h_1, h_2, \ldots, h_k)$ when $k \leq r$. Obviously, in this case not all of the operators I_r^n for $r = 1, 2, \ldots, n$ or $K_r^{p_n}$ for $r = 1, 2, \ldots, n$ are independent, but we may easily choose an independent set by letting $r = 1, 2, \ldots, k$. However, we must use an over complete set of operators to specify the IR $[u]$ since we have no way of determining beforehand how many rows are contained in $[u]$. Unfortunately, for $r > k$ it is possible that $p_n < r$, in which case $K_r^{p_n}$ is not defined. We may remedy this situation and use the set of operators $K_r^{p_n}$ for $r = 1, 2, \ldots, n$ to specify any $[u]$ if we stipuate that

$$K_r^{p_n} = 0, \quad \text{when } p_n < r. \tag{2.19}$$

We have shown that the IR's of U(l) may be specified by either the operators I_r^l or $K_r^{p_l}$ for $r = 1, 2, \ldots, l$. Since both sets of operators are complete and independent, we may expand one set in terms of the other.[41,42]

It follows that since the Gel'fand operators I_r^l are mutually commuting invariant Hermitian operators of U(n) for $r = 1, 2, \ldots, l$, and $l = 1, 2, \ldots, n$, so are the operators:

$$\begin{matrix} K_n^{p_n} K_{n-1}^{p_n} \cdots K_1^{p_n} \\ K_{n-1}^{p_{n-1}} \cdots K_1^{p_{n-1}} \\ \ddots \\ K_1^{p_1} \end{matrix}$$

where $K_r^{p_l} = 0$ for $p_l < r$.

Thus the set of eigenvalues of the r-cycle class operators above completely specify a canonical or Gel'fand basis of U(n). Also, because of the relation (1.30), such a set of eigenvalues will also specify the canonical basis of SU(n).

III. CANONICAL WEYL BASES

Let $|\phi_i^l\rangle$ be the ith state of the lth particle which is taken as the ith fundamental basis of U(n) or SU(n). These bases are assumed to be orthogonal,

$$\langle \phi_i^l | \phi_{i'}^{l'}\rangle = \delta_{ii'}\delta_{ll'}. \tag{3.1}$$

The single particle generators e_{ij}^l of the fundamental representation of U(n) operate on the mth particle state such that

$$e_{ij}^l |\phi_k^m\rangle = \delta_{lm}\delta_{jk}|\phi_i^l\rangle. \tag{3.2}$$

Hence, the e_{ij}^l obey the commutation relations

$$[e_{ij}^m, e_{kl}^n] = \delta_{mn}(e_{il}^m\delta_{jk} - e_{kj}^m\delta_{il}), \tag{3.3}$$

as required.

The pth direct product of the fundamental bases,

$$|\phi_{(i)}\rangle \equiv |\phi_{i_1}^1\rangle|\phi_{i_2}^2\rangle\cdots|\phi_{i_p}^p\rangle = |\phi_{i_1}^1\phi_{i_2}^2\cdots\phi_{i_p}^p\rangle,$$

for $i_1, i_2, \ldots, i_p = 1, 2, \ldots, n$ is an n-dimensional, p-rank tensor space and forms a reducible basis of U(n) with generators

$$E_{ij} = \sum_{l=1}^{p} e_{ij}^l. \tag{3.4}$$

These generators obey the required commutation relations (1.24). The tensor space also forms an orthonormal basis,

$$\langle\phi_{(i)}|\phi_{(j)}\rangle = \delta_{i_1j_1}\delta_{i_2j_2}\cdots\delta_{i_pj_p} = \delta_{(i)(j)}. \tag{3.5}$$

We may define two commuting groups of permutation operators $[q]$ and (q) which act on our tensor space. A permutation $[q]$ acts on the state labels associated with each particle and transfers them to other particles. Thus the *particle* permutation [123] means transfer the state label of particle 1 to particle 2, transfer the state label of particle 2 to particle 3, and transfer the state label of particle 3 to particle 1. For example,

$$[123]|\phi_{i_1}^1\phi_{i_3}^2\phi_{i_2}^3\rangle = |\phi_{i_2}^1\phi_{i_1}^2\phi_{i_3}^3\rangle.$$

A permutation (q) acts on the state labels themselves and transfers them to other state labels. Thus the *state* permutation (123) means transfer state label i_1 to i_2, transfer state label i_2 to i_3, and transfer state label i_3 to i_1. For example,

$$(123)\,|\phi^1_{i_1}\phi^2_{i_3}\phi^3_{i_2}\rangle = |\phi^1_{i_3}\phi^2_{i_2}\phi^3_{i_1}\rangle.$$

Because of the different multiplicative properties of $[q]$ and (q) their irreducible matrix representations are different and obey the relation

$$D^{[u]}_{ij}(q) = D^{[u]}_{ji}[q]. \tag{3.6}$$

We also note that

$$(q)\,|\phi_{(i)}\rangle = [q]\,|\phi_{(i)}\rangle. \tag{3.7}$$

When operating on the tensor space, the generators E_{ij} do not affect the ordering of the subscripts so that the E_{ij} commute with all $[q]$ and (q), i.e.,

$$E_{ij}[q] = [q]\,E_{ij}, \tag{3.8a}$$

$$E_{ij}(q) = (q)\,E_{ij}. \tag{3.8b}$$

We now reduce the tensor space under permutations of S_p. Let $[P^{[u]}_{rs}]$ operate with particle permutations and $(P^{[u]}_{rs})$ operate with state permutations. From (1.10) we have that the projected basis $[P^{[u]}_{rs}]\,|\phi_{(i)}\rangle$ transforms like an irreducible basis $|{}^{[u]}_{r}\rangle$ of S_p under particle permutations $[q]$,

$$[q][P^{[u]}_{rs}]\,|\phi_{(i)}\rangle = \sum_{r'} [P^{[u]}_{r's}]\,|\phi_{(i)}\rangle D^{[u]}_{r'r}[q]. \tag{3.9}$$

Also from (3.6), (1.11), and (3.7) we have

$$\begin{aligned}(q)[P^{[u]}_{rs}]\,|\phi_{(i)}\rangle &= [P^{[u]}_{rs}](q)\,|\phi_{(i)}\rangle \\ &= [P^{[u]}_{rs}][q]\,|\phi_{(i)}\rangle \\ &= \sum_{s'} [P^{[u]}_{rs'}]\,|\phi_{(i)}\rangle D^{[u]}_{ss'}[q] \\ &= \sum_{s'} [P^{[u]}_{rs'}]\,|\phi_{(i)}\rangle D^{[u]}_{s's}(q).\end{aligned} \tag{3.10}$$

Thus $[P^{[u]}_{rs}]\,|\phi_{(i)}\rangle$ transforms like an irreducible basis $|{}^{[u]}_{s}\rangle$ of S_p under state permutations (q). We see that the particle projection operator reduces the tensor space under both particle and state permutations. Also from (3.6) and (3.7) we have

$$[P^{[u]}_{rs}]\,|\phi_{(i)}\rangle = (P^{[u]}_{sr})\,|\phi_{(i)}\rangle.$$

One may use either particle or state projection operators to reduce the space. We choose to use particle projection operators and from now on we assume the $P^{[u]}_{rs}$ are *particle* projection operators unless otherwise denoted.

Let

$$|{}^{[u]}_{s\,r}\rangle = P^{[u]}_{rs}\,|\phi_{(i)}\rangle. \tag{3.11}$$

The different irreducible bases $|{}^{[u]}_{s\,r}\rangle$ of S_p for a fixed s and (i) are orthogonal. To show this, we first note that the particle permutations are unitary operators on the tensor space:

$$([q]\,|\phi_{(i)}\rangle)^\dagger\,|\phi_{(j)}\rangle = \langle\phi_{(i)}\,|[q^{-1}]|\,\phi_{(j)}\rangle,$$

or

$$[q]^\dagger = [q^{-1}]. \tag{3.12}$$

Using (3.12), (1.8), and (1.9) we have

$$[P^{[u]}_{rs}]^\dagger = [P^{[u]}_{sr}]. \tag{3.13}$$

From the above equation it follows that

$$\begin{aligned}\langle{}^{[u]}_{s\,r}|{}^{[u']}_{s'r'}\rangle &= \langle\phi_{(i)}\,|P^{[u]\dagger}_{rs}P^{[u']}_{r's'}|\,\phi_{(i')}\rangle \\ &= \langle\phi_{(i)}\,|P^{[u]}_{sr}P^{[u']}_{r's'}|\,\phi_{(i')}\rangle \\ &= \delta_{[u][u']}\,\delta_{rr'}\langle\phi_{(i)}\,|P^{[u]}_{ss'}|\,\phi_{(i')}\rangle.\end{aligned} \tag{3.14}$$

The number of times the IR $[u]$ of S_p is contained in the tensor space is the number of independent bases $|{}^{[u]}_{s\,r}\rangle$ for a given $[u]$ and r. This will be shown to be equal to $f^{[u]}$.

The particle projection operator also reduces the tensor space under transformations of the generators E_{ij} of U(n). We have

$$\begin{aligned}[q]E_{ij}\,|{}^{[u]}_{s\,r}\rangle &= E_{ij}[q]\,|{}^{[u]}_{s\,r}\rangle \\ &= \sum_{r'} E_{ij}\,|{}^{[u]}_{s\,r'}\rangle D^{[u]}_{r'r}[q],\end{aligned}$$

so that bases with fixed r transform among themselves under all E_{ij} of U(n) and form a representation of U(n). The dimension of this representation of U(n) for a given r and $[u]$ is the number of independent bases, $f^{[u]}$. The number of such representations contained in the tensor space is the number of independent bases $|{}^{[u]}_{s\,r}\rangle$ for a given $[u]$, s, and (i). But this is just the dimension of the IR $[u]$ of S_p, or $l^{[u]}$.

To complete the reduction of S_p and U(n) on the tensor space using particle projection operators one can determine the $f^{[u]}$ independent bases for fixed $[u]$ and r and then orthonormalize these using the standard Graham–Schmidt procedures. An alternate method, which we shall adopt, is to simultaneously diagonalize the mutually commuting Hermitian invariant operators of U(n),

$$\begin{matrix}K^{p_n}_n\,K^{p_n}_{n-1}\cdots K^{p_n}_1 \\ K^{p_{n-1}}_{n-1}\cdots K^{p_{n-1}}_1 \\ \ddots\quad\vdots \\ K^{p_1}_1\end{matrix},$$

where $K^{p_l}_r = 0$ when $p_l < r$, in our bases $|{}^{[u]}_{s\,r}\rangle$. The eigenvalues of the operators must uniquely specify the canonical irreducible basis of U(n) to which the eigenvectors correspond. Hence, bases with the same set of eigenvalues must be equal within a normalization factor. Because of the Hermitian properties of the invariant operators, bases with different sets of eigenvalues must be orthogonal and correspond to different canonical irreducible bases of U(n).

An irreducible basis constructed in this manner is a Weyl basis of U(n). Because of our choice of operators the Weyl basis will also be a canonical or Gel'fand basis of U(n).

We can reduce our work by eliminating the bases which are obviously not independent. Denote by $|\phi_\lambda\rangle$ any tensor with state labels consisting of λ_1 ones, λ_2 twos, ..., and λ_n n's in the subscripts. Using (3.10) in the following form:

$$P^{[u]}_{rs}[q]\,|\phi_\lambda\rangle = \sum_{s'} P^{[u]}_{rs'}\,|\phi_\lambda\rangle D^{[u]}_{ss'}[q],$$

we may let $|\phi_\lambda\rangle$ have any order in the individual particle states and still obtain the same independent bases. For this reason we may choose the ordering below,

$$|\phi_\lambda\rangle = |\phi^1_1\phi^2_1\cdots\phi^{p_1}_1\phi^{p_1+1}_2\cdots\phi^{p_2}_2\quad\cdots\phi^{p_{n-1}+1}_n\cdots\phi^{p_n}_n\rangle, \tag{3.15}$$

where $\sum_{i=1}^{r} \lambda_i = p_r$, and $p_n = p$, without losing generality. We now define

$$\left|{}^{[u]}_{s\,r}\right\rangle = P^{[u]}_{rs} \left| \phi_\lambda \right\rangle, \tag{3.16}$$

and note that

$$E_{ii} \left|{}^{[u]}_{s\,r}\right\rangle = \lambda_i \left|{}^{[u]}_{s\,r}\right\rangle. \tag{3.17}$$

The weight of the basis $|{}^{[u]}_{s\,r}\rangle$ is $(\lambda) = (\lambda_1 \lambda_2 \cdots \lambda_n)$. The bases $|{}^{[u]}_{s\,r}\rangle$ are already eigenvectors of the invariant operators $K_1^{p_r}$ for $r = 1, 2, \ldots, n$, where

$$K_1^{p_r} \left|{}^{[u]}_{s\,r}\right\rangle = p_r \left|{}^{[u]}_{s\,r}\right\rangle, \tag{3.18}$$

so bases with different weights must be orthogonal. This is also obvious from (3.5).

The question now arises as to which permutation convention to use for the class operators $K_r^{p_l}$ when operating on the tensor space bases. Because the state operators (q) and the generators E_{ij} obey the same commutation relations with respect to the particle projection operators, we use the state classes $(K_r^{p_l})$. Hence, the invariant operators I_k^l are expanded in terms of state class operators $(K_r^{p_l})$ for $r = 1, 2, \ldots, k$ when acting on the tensor space. From now on we assume the $K_r^{p_l}$ are *state* class operators unless otherwise denoted. Since the I_k^l act only on states with state numbers $1, 2, \ldots, l$, the $K_r^{p_l}$ must act only on state labels with state numbers $1, 2, \ldots, l$. We define $S_{(p_l)}$ to be the subgroup of S_p corresponding to permutations of the state labels of our tensor bases with state numbers $1, 2, \ldots, l$ so that $K_r^{p_l} \subset S_{(p_l)}$. For example, if $i_1 i_2 i_3 i_4 i_5 = 13243$, then $S_{(p_3)} = S_{(4)}$ is the group of permutations of i_1, i_2, i_3, and i_5. Note that $S_{(4)}$ differs from S_4 where the latter is the group of permutations of the *first* four state labels i_1, i_2, i_3, and i_4.

We may now show directly that the class operators $K_r^{p_l}$ for $l = 1, 2, \ldots, n$ and $r = 1, 2, \ldots, l$ are mutually commuting invariant Hermitian operators of $\mathrm{U}(n)$. We have

$$[K_k^{p_m}, K_r^{p_l}] = 0, \tag{3.19}$$

since a class of a group commutes with all elements of that group and one of the groups $S_{(p_m)}$ or $S_{(p_l)}$ is a subgroup of the other. From (3.8b) it follows that the r-cycle class operators are invariant operators of $\mathrm{U}(n)$:

$$[K_r^{p_l}, E_{ij}] = 0. \tag{3.20}$$

If a class contains the element (q) it also contains $(q)^{-1}$. If $(K_r^{p_l})^{-1}$ is the inverse of all the permutation terms in $(K_r^{p_l})$, then

$$(K_r^{p_l})^\dagger = (K_r^{p_l})^{-1} = (K_r^{p_l}), \tag{3.21}$$

i.e., the r-cycle class operators are Hermitian.

The bases $|{}^{[u]}_{s\,r}\rangle$ are already eigenvectors of the operators $K_1^{p_l}$ for $l = 1, 2, \ldots, n$ with eigenvalues p_l. As stated before, it is just these eigenvalues that determine which canonical scheme of r-cycle class operators to use, i.e., which subgroup $S_{(p_l)}$ of S_p the r-cycle class operators act on.

We now prove that the projected tensor bases $|{}^{[u]}_{s\,r}\rangle$ are eigenvectors of $K_k^{p_n}$ for $k = 1, 2, \ldots, n$. From (3.10) we have that $|{}^{[u]}_{s\,r}\rangle$ transforms like an irreducible basis $|{}^{[u]}_{s}\rangle$ under permutations (q) of S_p. Then from (1.17) we have

$$K_k^{p_n} \left|{}^{[u]}_{s\,r}\right\rangle = \frac{N_k^{p_n} \chi_k^{[u]}}{l^{[u]}} \left|{}^{[u]}_{s\,r}\right\rangle \tag{3.22}$$

for $k = 2, 3, \ldots, n$ where $N_k^{p_n}$ is the order of the class $K_k^{p_n}$. This proves that the independent bases $|{}^{[u]}_{s\,r}\rangle$ for fixed r form an irreducible basis of $\mathrm{U}(n)$ corresponding to partition $[u]$ if complete.

We now let the projection operators $P^{[u]}_{rs}$ be expanded in terms of the *canonical* IR of S_p. These $P^{[u]}_{rs}$ will then be proportional to the semi-normal projection operators $O^{[u]}_{rs}$. In this case we have from (3.10) that $|{}^{[u]}_{s\,r}\rangle$ transforms like the *canonical* irreducible basis $|{}^{[u]}_{(s)}\rangle$ associated with the standard tableau $T^{[u]}_{(s)}$ under permutations (q) of S_p.

The basis $|{}^{[u]}_{s\,r}\rangle$ therefore transforms like a canonical irreducible basis $|{}^{[u]^{p_l}}_{(s)}\rangle$ of $[u]^{p_l}_s$ under permutations (q) of S_{p_l}. For $K_r^{p_l}$ to be a class operator of the subgroup S_{p_l} in the canonical reduction of the basis $|{}^{[u]}_{(s)}\rangle$, it is necessary that the subgroup $S_{(p_l)}$ of $K_r^{p_l}$ correspond to permutations of the *first* p_l state labels. Since $K_r^{p_l}$ permutes the state numbers $1, 2, \ldots, l$, for $K_r^{p_l}$ to be a class of S_{p_l} in this canonical reduction these state numbers must be in the first state labels $i_1, i_2, \ldots, i_{p_l}$. This is the reason for our particular choice of $|\phi_\lambda\rangle$ with definite order such that $i_1 \leq i_2 \leq \cdots \leq i_p$. From the above considerations, we have

$$K_k^{p_l} \left|{}^{[u]}_{s\,r}\right\rangle = \frac{N_k^{p_l} \chi_k^{[u]_s^{p_l}}}{l^{[u]_s^{p_l}}} \left|{}^{[u]}_{s\,r}\right\rangle \tag{3.23}$$

for $l = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, l$. Thus the canonically projected Weyl basis $|{}^{[u]}_{s\,r}\rangle$ transforms like an irreducible basis of $[u]_s^{p_l}$ under $\mathrm{U}(l)$. From (1.32) we see that if $|{}^{[M]}_{\langle m\rangle}\rangle$ is a canonical irreducible basis of $\mathrm{U}(n)$, then $|{}^{[u]}_{s\,r}\rangle = |{}^{[M]}_{\langle m\rangle}\rangle$ when

$$[u]_s^{p_l} = [m_{1l}\, m_{2l} \cdots m_{ll}] \quad \text{for } l = 1, 2, \ldots, n-1,$$
$$[u]_s^{p_n} = [u] = [M]. \tag{3.24}$$

Both (s) and weight $(\lambda) = (\lambda_1 \lambda_2 \cdots \lambda_n)$ where $\sum_{i=1}^{l} \lambda_i = p_l$ uniquely specify $\langle m \rangle$ for a given $[u] = [M]$.

We define a tableau of $\mathrm{U}(n)$, $T^{[u]}_{(s)} \phi_\lambda$, to be a partition $[u]$ with state label i_r in the box containing r of standard tableau $T^{[u]}_{(s)}$. For example,

$${}^{124}_{35}\, \phi_1^1 \phi_2^2 \phi_2^3 \phi_3^4 \phi_3^5 = {}^{123}_{23}.$$

Then $[u]_s^{p_l}$ is simply the partition remaining after removing state numbers $l+1, l+2, \ldots, n$ from $T^{[u]}_{(s)} \phi_\lambda$. We have a one to one correspondence between the tableaus $T^{[u]}_{(s)} \phi_\lambda$ and states $|{}^{[u]}_{s\,r}\rangle$.

Not every tableau $T^{[u]}_{(s)} \phi_\lambda$ of $\mathrm{U}(n)$ corresponds to a standard tableau of $\mathrm{U}(n)$, since the "betweeness conditions" (1.33),

$$m_{i\,l+1} \geq m_{il} \geq m_{i+1\, l+1},$$

where $[u]_s^{p_l} = [m_{1l}\, m_{2l} \cdots m_{ll}]$ are not necessarily satisfied for all l. However, if a tableau $T^{[u]}_{(s)} \phi_\lambda$ of $\mathrm{U}(n)$ contains no identical state numbers in a column, the "betweeness conditions" will always be satisfied and the resulting tableau of $\mathrm{U}(n)$ will correspond to a standard tableau $T^{[u]}_{(s)}$.

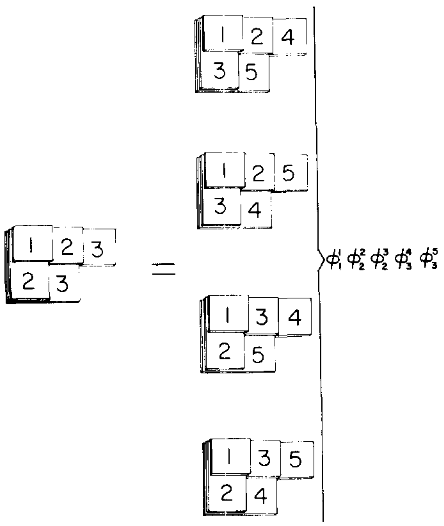

FIG. 6. Correspondence between standard tableaus of U(n) and S_p. The standard tableau of U(3) on the left corresponds to the standard tableaus of S_5 on the right when the numbers 1, 2, 3, 4, 5 are replaced by the state numbers 1, 2, 2, 3, 3 respectively. The first standard tableau of S_5 is derived from the standard tableau of U(3) by replacing the state numbers (1), (2), and (3) by the numbers (1), (2,3), and (4,5) respectively in "book order."

We have not yet shown that all standard tableaus $T^{[u]}_{\langle s\rangle}$ of U(n) for a given weight (λ) correspond to some tableau $T^{[u]}_{(s)}\phi_\lambda$, i.e., we have not yet shown that the projected bases $|{}^{[u]}_{s\,r}\rangle$ are complete. We need a way of replacing the numbers $1, 2, \ldots, n$ in the standard tableau $T^{[u]}_{\langle s\rangle}$ of U(n) with the nonrepeated numbers $1, 2, \ldots, p$ to produce a standard tableau $T^{[u]}_{(s)}$ of S_p such that $T^{[u]}_{\langle s\rangle} = T^{[u]}_{(s)}\phi_\lambda$. One such way is to simply replace the numbers i in $T^{[u]}_{\langle s\rangle}$ by the numbers $p_{i-1}+1, p_{i-1}+2, \ldots, p_i$ in "book order" such that they increase to the right in the rows and down in the columns. Usually there are *several* different standard tableaus $T^{[u]}_{(s)}$ of S_p which yield a given $T^{[u]}_{\langle s\rangle}$ of U(n) as shown in Fig. 6. It follows that the projected states $|{}^{[u]}_{s\,r}\rangle$ corresponding to *different* standard tableaus $T^{[u]}_{\langle s\rangle}$ form a complete and independent set of canonical bases of U(n). The number of such independent bases will be $f^{[u]}$ as indicated previously.

Since the invariant operators are Hermitian, eigenvectors belonging to different sets of eigenvalues are orthogonal. The eigenvalues of $K_1^{p_l}$ determine p_l and the eigenvalues of $K_r^{p_l}$ for $r = 2, 3, \ldots, l$ uniquely determine the partition $[u]_s^{p_l}$. So, if $[u]_s^{p_l} \neq [u]_{s'}^{p_l}$ for some $l = 1, 2, \ldots, n-1$, or correspondingly, if $T^{[u]}_{(s)}\phi_\lambda \neq T^{[u]}_{(s')}\phi_{\lambda'}$, then

$$\langle {}^{[u]}_{s\,r} | {}^{[u]}_{s'\,r}\rangle = 0.$$

Similarly, eigenvectors belonging to the same set of eigenvalues must be equal within a normalization factor. So, if $T^{[u]}_{(s)}\phi_\lambda = T^{[u]}_{(s')}\phi_{\lambda'}$, then

$$|{}^{[u]}_{s\,r}\rangle = C\,|{}^{[u]}_{s'\,r}\rangle,$$

and $\lambda = \lambda'$, where C is a constant. This result has been proven by Goddard using only the properties of the canonical IR's of S_p.[43]

Projected states with tableaus $T^{[u]}_{(s)}\phi_\lambda$ having identical state numbers in a column are orthogonal to projected states with standard tableaus and must therefore be null states. Again, this has been proven using only the properties of the canonical IR's of S_p.[44] As an example, we have

$$\left|{}^{1\,2\,4}_{3\,5}\ \phi_1^1\phi_1^2\phi_1^3\phi_3^4\phi_3^5\right\rangle = \left|{}^{1\,1\,3}_{1\,3}\right\rangle = 0.$$

Now if the IR $[u]$ of U(n) had more than n rows, $T^{[u]}_{(s)}\phi_\lambda$ would have at least two state numbers in a column for any λ and s. So $|{}^{[u]}_{s\,r}\rangle = 0$ for any Weyl basis of U(n) when $[u]$ has more than n rows.

Since the canonically projected nonnull states $|{}^{[u]}_{s\,r}\rangle$ form a basis $|{}^{[u]}_{(r)}\rangle$ under permutations (q) of S_p and a basis $|{}^{[u]}_{\langle s\rangle}\rangle$ under generators E_{ij} of U(n), we write

$$|{}^{[u]}_{s\,r}\rangle \equiv |{}^{[u]}_{\langle s\rangle(r)}\rangle.$$

A canonically projected Weyl basis simultaneously forms a canonical basis of S_p and a canonical basis, or Gel'fand basis, of U(n). We shall show the significance of the subspace of all such states $|{}^{[u]}_{\langle s\rangle(r)}\rangle$ more clearly in Paper II.

To normalize the canonical Weyl basis, we let $|{}^{[u]}_{\langle s\rangle}\rangle = N_s^{[u]}\,|{}^{[u]}_{\langle s\rangle(r)}\rangle$, where $\langle {}^{[u]}_{\langle s\rangle} | {}^{[u]}_{\langle s\rangle}\rangle = 1$. Then

$$(N_s^{[u]})^2 \langle {}^{[u]}_{\langle s\rangle(r)} | {}^{[u]}_{\langle s\rangle(r)}\rangle = (N_s^{[u]})^2 \langle \phi_\lambda | P^{[u]}_{ss} | \phi_\lambda\rangle = 1.$$

Finally, from (1.8) we have

$$N_s^{[u]} = \Big(p!/l^{[u]} \sum_{q\in S_\lambda} D^{[u]}_{ss}[q]\Big)^{1/2}, \tag{3.25}$$

where $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_n}$.

The canonical Weyl bases of U(n) are eigenvectors of the invariant operators of SU(n). So the canonical Weyl bases of U(n) are also canonical bases of SU(n). Thus if $|{}^{[u]}_{\langle s\rangle}\rangle = |{}^{[u]}_{\langle s\rangle(r)}\rangle$, we have that $|{}^{[u]}_{\langle s\rangle(r)}\rangle$ must also transform like $|{}^{[u']}_{\langle s\rangle}\rangle$ under SU(n), where

$$[u'] = [u_1 - u_n\ \ u_2 - u_n \cdots u_{n-1} - u_n\ \ 0]. \tag{3.26}$$

APPENDIX A[45]

We wish to find the eigenvalues $N_r^p \chi_r^{[u]}/l^{[u]}$ where $[u] = [u_1 u_2 \cdots u_m]$ is a partition labeling the IR's of S_p and satisfying the relation

$$\sum_{i=1}^{m} u_i = p. \tag{A1}$$

By partition we mean that the elements of $[u]$ are monotonically decreasing integers such that

$$u_1 \geq u_2 \geq \cdots \geq u_m. \tag{A2}$$

We first find the $[u]_{ir}$ for all i where

$$[u]_{ir} = [u_1 u_2 \cdots u_i - r \cdots u_m]. \tag{A3}$$

If $[u]_{ir}$ is not a partition, it may be possible to transform it into one using the following procedure. Let $[R] = [m-1\ m-2 \cdots 0]$ and find the permutation (q_i) such that $(q_i)([u]_{ir} + [R])$ has monotonically decreasing elements. Then find the $[u]'_{ir}$ for all i where

$$[u]'_{ir} = (q_i)([u]_{ir} + [R]) - [R]. \tag{A4}$$

Note that $[u]'_{ir}$ is not a partition if and only if $([u]_{ir} + [R])$ contains repeated or negative elements. Also note that $[u]'_{ir} = [u]_{ir}$ if $[u]_{ir}$ is a partition. We shall need only the $[u]'_{ir}$ which are partitions.[46]

We may now find the eigenvalues of the r-cycle class operators by using the simple hook-length formula below,

$$N_r^p \chi_r^{[u]}/l^{[u]} = \frac{1}{r} \sum_{i=1}^{m} \epsilon_{(q_i)} \frac{H([u])}{H([u]'_{ir})}. \tag{A5}$$

The sum in this equation is over all i such that $[u]'_{ir}$ is a partition. Also $\epsilon_{(q_i)}$ is 1 or -1 if the permutation (q_i) is an even or odd number of bicycles respectively. $H([u])$ is the product of hook-lengths of partition $[u]$ which has been presented in Fig. 1.

As an example, we find the eigenvalue of K_3^9 for IR $[u]=[432]$ of S_9. The $[u]_{ir}$ and corresponding $[u]'_{ir}$ are:

$$[432]_{13}=[132],$$

$$[432]_{23}=[402],$$

$$[432]_{33}=[43-1],$$

$$[432]'_{13}=(12)[342]-[210]=[222],$$

$$[432]'_{23}=(23)[612]-[210]=[411],$$

$$[432]'_{33}=[64-1]-[210]=[43-1].$$

Using the partitions $[432]'_{13}$ and $[432]'_{23}$ it is now a simple matter to evaluate the eigenvalue of K_3^9 in terms of hook-lengths as shown below,

$$\frac{N_3^9 \chi_3^{[432]}}{l^{[432]}} = \frac{1}{3}\left[-\frac{\begin{array}{cccc} 6 & 5 & 3 & 1 \\ 4 & 3 & 1 & \\ 2 & 1 & & \end{array}}{\begin{array}{ccc} & 4 & 3 \\ & 3 & 2 \\ & 2 & 1 \end{array}} - \frac{\begin{array}{cccc} 6 & 5 & 3 & 1 \\ 4 & 3 & 1 & \\ 2 & 1 & & \end{array}}{\begin{array}{cccc} 6 & 3 & 2 & 1 \\ 2 & & & \\ 1 & & & \end{array}} \right]$$

$$= -15.$$

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